The Leibniz Rule

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These notes are based on Harron 2006.

Theorem 1. *Suppose $f(x, y)$ is a function on the rectangle $R = [a, b] \times [c, d]$ and $\frac{\partial f}{\partial y}$ is continuous on R. Then*

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_a^b f(x, y) dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx \tag{1}$$

Proof. Consider the double integral,

$$\int_c^y \int_a^b \frac{\partial f}{\partial z}(x, z) dx dz$$

By interchanging the order of integrals and taking derivative we get the following,

$$\frac{\mathrm{d}}{\mathrm{d}y} \left(\int_c^y \int_a^b \frac{\partial f}{\partial z}(x, z) dx dz \right) = \frac{\mathrm{d}}{\mathrm{d}y} \left(\int_a^b \int_c^y \frac{\partial f}{\partial z}(x, z) dx dz \right) \tag{2}$$

By the Fundamental theorem of calculus, if $F' = f$ is continuous,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t f(x) dx \right) = f(t)$$

Then LHS of (2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}y} \left(\int_c^y \int_a^b \frac{\partial f}{\partial z}(x, z) dx dz \right) = \int_a^b \frac{\partial f}{\partial y}(x, y) dx$$

The RHS of (2) can be simplified as,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y} \left(\int_a^b \int_c^y \frac{\partial f}{\partial z}(x, z) dx dz \right) &= \frac{\mathrm{d}}{\mathrm{d}y} \left(\int_a^b f(x, y) dx - \int_a^b f(x, c) dx \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} \int_a^b f(x, y) dx \end{aligned}$$

which is the identity we set out to prove. □

Lemma 1. *The above theorem can be generalized as follows,*

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) dx = \int_{\phi_1(y)}^{\phi_2(y)} \frac{\partial f}{\partial y}(x, y) dx + \frac{\mathrm{d}\phi_2(y)}{\mathrm{d}y} f(x, \phi_2(y)) - \frac{\mathrm{d}\phi_1(y)}{\mathrm{d}y} f(x, \phi_1(y)) \tag{3}$$

Proof. Let $u = \phi_1(y), v = \phi_2(y), w = y$, then define $G(u, v, w)$ such that,

$$\int_{\phi_1(y)}^{\phi_2(y)} f(x, y)dx = \int_u^v f(x, w)dx \triangleq G(u, v, w)$$

By chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}y}G = \frac{\partial G}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\partial G}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\partial G}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}y} \tag{4}$$

Then,

$$\frac{\partial G}{\partial u} = \frac{\partial}{\partial u}\int_u^v f(x, w)dx = -f(u, w)$$
$$\frac{\partial G}{\partial v} = \frac{\partial}{\partial v}\int_u^v f(x, w)dx = f(v, w)$$
$$\frac{\partial G}{\partial w} = \frac{\partial}{\partial w}\int_u^v f(x, w)dx = \int_u^v \frac{\partial f}{\partial w}(x, w)dx$$

where the last equation follows from Theorem 1. □

References

Harron, Rob (2006). *MAT-203: The Leibniz Rule.* Online; accessed on March 01, 2016. URL: http://math.hawaii.edu/~rharron/teaching/MAT203/LeibnizRule.pdf.